The General Solutions of Linear ODE and Riccati Equation by Integral Series☆

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Abstract

This paper gives out the general solutions of variable coefficients Linear ODE and Riccati equation by way of integral series $\mathcal{E}(X)$ and $\mathcal{F}(X)$. Such kinds of integral series are the generalized form of exponential function, and keep the properties of convergent and reversible.

Keywords:
Linear ODE,Riccati equation,integral series, general solution,variable coefficients

1. Introduction

It is a classical problem to solve the n-th order Linear ODE :

$$\frac{d^n}{dx^n}u + a_1(x)\frac{d^{n-1}}{dx^{n-1}}u + a_2(x)\frac{d^{n-2}}{dx^{n-2}}u + \cdots + a_n(x)u = f(x) \tag{1}$$

which is equivalent to

$$\frac{d}{dx}U = AU + F \tag{2}$$

with

$$\begin{cases} U = \begin{bmatrix} \frac{d^{n-1}}{dx^{n-1}}u & \frac{d^{n-2}}{dx^{n-2}}u & \cdots & u \end{bmatrix}^T \\ F = \begin{bmatrix} f(x) & 0 & \cdots & 0 \end{bmatrix}^T \\ A(x) = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \end{cases} \tag{3}$$

As we all known,

1. if $\{a_n(x)\}$ are all constants, Eq.(1) could be solved by method of eigenvalue (Euler), or by exponential function in matrix form
$$U = e^{A\cdot x} \cdot C + e^{A\cdot x} \cdot \int_0^x e^{-A\cdot s} \cdot F(s)\, ds$$
where C *is a* $n \times 1$ *constant matrix* .

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2. if $\{a_n(x)\}$ are some variable coefficients, such as some special functions [Wang, P337,206]

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{n^2}{x^2})y = 0 \tag{Bessel Equation}$$

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{Legendre Equation}$$

special function theory answers them.

But when it comes to the general circumstances, the existing methods meet difficulties in dealing with Eq.(2) , because of the variable coefficients. In order to overcome it, two functions are invited :

1.1. Definition

$$\begin{cases} \mathcal{E}[X(x)] = I + \int_0^x X(t)\,dt + \int_0^x X(t)\int_0^t X(s)\,dsdt + \int_0^x X(t)\int_0^t X(s)\int_0^s X(\xi)\,d\xi dsdt + \cdots \\ \mathcal{F}[X(x)] = I + \int_0^x X(t)\,dt + \int_0^x \int_0^t X(s)\,dsX(t)\,dt + \int_0^x \int_0^t \int_0^s X(\xi)\,d\xi\; X(s)ds\; X(t)\,dt + \cdots \end{cases} \tag{4}$$

It will be seen that such definition is reasonable and necessary. Clearly, when $X(x)$ and $\int_0^x X(t)dt$ are exchangeable, then

$$\mathcal{E}[X(x)] = e^{\int_0^x X(t)dt} = \mathcal{F}[X(x)]$$

Besides, $\mathcal{E}(X)$ and $\mathcal{F}(X)$ extend some main properties of the exponential functions, such as convergent , reversible and determinant (see Theorem 3.1). In addition, a $n \times m$ matrix $A(x) = (a_{ij}(x))_{nm}$ is bounded and integral in [0,b] means that all its element $a_{ij}(x)$ are bounded and integral in [0,b].

2. Main Results

Theorem 2.1. *the general solution of the Linear ODE (2) is:*

$$U = \mathcal{E}[A(x)] \cdot C + \mathcal{E}[A(x)] \cdot \int_0^x \mathcal{F}[-A(s)] \cdot F(s)\,ds \tag{5}$$

where C is a $n \times 1$ constant matrix .

Theorem 2.2. *For the bounded and integrable matrix , $A(x) = (a_{ij})_{nn}$, $B(x) = (b_{ij})_{mm}$, $P(x) = (p_{ij})_{mn}$, $Q(x) = (q_{ij})_{nm}$, in [0,b], the general solution of Riccati equation*

$$\frac{d}{dx}W + WPW + WB - AW - Q = 0 \tag{6}$$

is

$$W = W_1 \cdot W_2^{-1} \tag{7}$$

where

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathcal{E}\left(\begin{bmatrix} A & Q \\ P & B \end{bmatrix}\right) \cdot \begin{bmatrix} W|_{x=0} \\ I \end{bmatrix} \tag{8}$$

or the other equivalent form:

$$W = U_2^{-1} \cdot U_1 \tag{9}$$

where

$$\begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} I & W|_{x=0} \end{bmatrix} \cdot \mathcal{F}\left(\begin{bmatrix} -B & P \\ Q & -A \end{bmatrix}\right) \tag{10}$$

3. Solutions of Linear ODE

3.1. Properties of $\mathcal{E}(X)$ and $\mathcal{F}(X)$

From the Definition(4), it holds that

$$\begin{cases} \dfrac{d}{dx}\mathcal{E}\Big[X(x)\Big] = X\cdot\mathcal{E}\Big[X(x)\Big] \\ \dfrac{d}{dx}\mathcal{F}\Big[X(x)\Big] = \mathcal{F}\Big[X(x)\Big]\cdot X \end{cases} \tag{11}$$

Now, we will see more explicit properties of $\mathcal{E}(X)$ and $\mathcal{F}(X)$.

Theorem 3.1 (Properties of $\mathcal{E}(X)$ and $\mathcal{F}(X)$). *If $X(x)$ is bounded and integrable, it holds that*

1. *$\mathcal{E}(X)$ and $\mathcal{F}(X)$ are convergent;*
2. $$\det\mathcal{E}(X) = \det\mathcal{F}(X) = \det e^{\int_0^x X(t)dt} = e^{\int_0^x trX(t)dt} = e^{tr\int_0^x X(t)dt} \tag{12}$$
3. *$\mathcal{E}(X)$ and $\mathcal{F}(X)$ are reversible, and*
$$\mathcal{F}(X)\mathcal{E}(-X) = \mathcal{E}(-X)\mathcal{F}(X) = I \tag{13}$$

Proof. 1. Firstly, $\mathcal{E}(A)$ is convergent,since $\{a_k(x)\}_{k=1}^n$ are bounded in [0,b]:

$$\exists M>0, \text{ s.t. } |a_k(x)|<M, \qquad \forall x\in[0,b], \qquad k=1,2,\cdots,n$$

So

(a)

$$\Big\|\int_0^x A(t)\,dt\Big\| = max\left|\int_0^x a_k(t)\,dt\right| < M|x|$$

(b)

$$\Big\|\int_0^x A(t)\int_0^t A(s)\,dsdt\Big\| = max\left|\sum_i\int_0^x a_k(t)\int_0^t a_i(s)\,dsdt\right| < nM^2\left|\int_0^x\int_0^t 1dsdt\right| < \tfrac{n}{2!}(M|x|)^2$$

(c)

$$\begin{aligned} \Big\|\int_0^x A(t)\int_0^t A(s)\int_0^s A(\xi)\,d\xi dsdt\Big\| &= max\left|\sum_{i,j}\int_0^x a_k(t)\int_0^t a_i(s)\int_0^s a_j(\xi)\,d\xi dsdt\right| \\ < \qquad n^2M^3\left|\int_0^x\int_0^t\int_0^s d\xi dsdt\right| &< \frac{n^2}{3!}(M|x|)^3 \end{aligned}$$

(d) $\cdots\cdots$

It follows that

$$\|\mathcal{E}(A)\| \quad < \quad 1+\frac{1}{n}\Big[nM|x| + \frac{1}{2!}(nMx)^2 + \frac{1}{3!}(nMx)^3 + \cdots\Big] = 1+\frac{1}{n}e^{nM|x|}$$

Clearly, $\mathcal{E}(A)$ is convergent.

Similarly, $\mathcal{F}(X)$ is also convergent.

2. $\forall n \times n$ *matrix* $A(x)$*, if* $trA(x)$ *is bounded and integral , then*

$$\det \mathcal{E}(A(x)) = e^{\int_0^x trA(t)dt} = e^{tr \int_0^x A(t)dt} \tag{14}$$

which is a special case of Abel's formula[Chen]: *If W and B are* $n \times n$ *matrixes , s.t.*

$$\frac{d}{dx} W = BW \tag{15}$$

then,

$$\det W = e^{trB} \tag{16}$$

Here we just take 2×2 matrix for verification:

Let $Y(x) = \mathcal{E}\Big[A(x)\Big] = \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix}, A(x) = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$

so $\frac{d}{dx} Y = A \cdot Y$means that

$$\frac{d}{dx} \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \tag{17}$$

it follows

$$\begin{aligned} \frac{d}{dx}(\det Y) &= \det \begin{bmatrix} \frac{d}{dx} y_{1,1} & \frac{d}{dx} y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} + \det \begin{bmatrix} y_{1,1} & y_{1,2} \\ \frac{d}{dx} y_{2,1} & \frac{d}{dx} y_{2,2} \end{bmatrix} \\ &= det \begin{bmatrix} a_{1,1}y_{1,1} + a_{1,2}y_{2,1} & a_{1,1}y_{1,2} + a_{1,2}y_{2,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} + \det \begin{bmatrix} y_{1,1} & y_{1,2} \\ a_{2,1}y_{1,1} + a_{2,2}y_{2,1} & a_{2,1}y_{1,2} + a_{2,2}y_{2,2} \end{bmatrix} \\ &= a_{1,1} \det \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} + a_{2,2} \det \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \\ &= \Big[a_{1,1} + a_{2,2}\Big] detY = trA \cdot detY \end{aligned}$$

Thus, Abel's formula holds and $\mathcal{E}(A(x))$ is reversible.

By the times:

$$\det \mathcal{F}(X) = e^{\int_0^x trX(t)dt} = e^{tr \int_0^x X(t)dt} \tag{18}$$

so, all we need to proof is

$$\det e^{\int_0^x X(t)dt} = e^{\int_0^x trX(t)dt} \tag{19}$$

Because $e^{\int_0^x X(t)dt}$ no longer satisfies Abel's formula (one reason is X and $\int_0^x X(t)dt$ are unnecessarily exchangeable) , we seek the other approach:

$\forall n \times n$ matrix A, $\exists n \times n$ reversible matrix P , s.t.

$$P^{-1}AP = diag\{J_1, J_2, \cdots, J_s\} := J$$

J is A's Jordan matrix, J_i *is the Jordan block with eigenvalue* $\lambda_i(x)$*.*

It follows that

$$e^{J_i} = e^{\lambda_i(x)} \begin{bmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \cdots \\ 0 & 1 & 1 & \frac{1}{2!} & \cdots & \cdots \\ 0 & 0 & 1 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{20}$$

So,

$$P^{-1}e^{A}P = e^{P^{-1}AP} = e^{J} = diag\{e^{J_1}, e^{J_2}, \cdots, e^{J_s}\}$$

Therefore

$$\det e^{A} = \det e^{J} = e^{trJ} = e^{trA}$$

which yields

$$\det e^{\int_0^x X(t)dt} = e^{\int_0^x trX(t)dt}$$

3. *Notice that $\forall n \times n$ matrix A, there exists a companion matrix A^*,s.t.*

$$A \cdot A^* = A^* \cdot A = \mathtt{det}A \cdot I \tag{21}$$

so, if $\det A \neq 0$ *, A is invertible.*

Therefore, $\mathcal{E}(X)$ and $\mathcal{F}(X)$ are invertible.
Furthermore, it holds that

$$\mathcal{F}(X)\mathcal{E}(-X) = \mathcal{E}(-X)\mathcal{F}(X) = I \tag{22}$$

Because:

(a)

$$\frac{d}{dx}\Big[\mathcal{F}(X)\mathcal{E}(-X)\Big] = \frac{d}{dx}\mathcal{F}(X) \cdot \mathcal{E}(-X) + \mathcal{F}(X) \cdot \frac{d}{dx}\mathcal{E}(-X) = \mathcal{F}(X)X \cdot \mathcal{E}(-X) - \mathcal{F}(X) \cdot X\mathcal{E}(-X) = 0$$

So,

$$\mathcal{F}(X)\mathcal{E}(-X) = const. = [\mathcal{F}(X)\mathcal{E}(-X)]\big|_{x=0} = I$$

(b) Due to the special property(21) of matrix, Eq.(22) is obtained.

□

3.2. *Proof of Theorem2.1*

Proof. According to Definition(4) and Theorem 3.1 , it follows

$$\begin{cases} \dfrac{d}{dx}\mathcal{E}[A(x)] = & A(x) \cdot \mathcal{E}[A(x)] \\ \dfrac{d}{dx}G(x) = & A(x) \cdot G(x) + F \end{cases} \tag{23}$$

where

$$G(x) = \mathcal{E}[A(x)] \cdot \int_0^x \mathcal{F}[-A(s)] \cdot F(s)\, ds$$

because

$$\begin{aligned} \frac{d}{dx}G(x) =& \frac{d}{dx}\mathcal{E}[A(x)] \cdot \int_0^x \mathcal{F}[-A(s)] \cdot F(s)\, ds + \mathcal{E}[A(x)] \cdot \mathcal{F}[-A(x)] \cdot F(x) \\ =& A(x) \cdot \mathcal{E}[A(x)] \cdot \int_0^x \mathcal{F}[-A(s)] \cdot F(s)\, ds + F \end{aligned}$$

Clearly $U(x) = \mathcal{E}[A(x)] \cdot C + \mathcal{E}[A(x)] \cdot \int_0^x \mathcal{F}[-A(s)] \cdot F(s)\, ds$ is convergent.

Moreover, since $\mathcal{E}(A)$ is reversible, $U(x)$ is the general solution of Eq.(2).
□

Theorem 3.2. *Assume that $A(x) = (a_{ij})_{n\times n}$, $B(x) = (b_{ij})_{m\times m}$, $P(x) = (p_{ij})_{n\times m}$ are bounded and integrable matrixes , and $U(x)$ is the desired $n \times m$ matrix. The Linear ODE :*

$$\frac{d}{dx}U = A(x)U + UB(x) + P(x) \tag{24}$$

has general solutions

$$U(x) = \mathcal{E}(A)\Big[\int_0^x \mathcal{F}(-A(t))P(t)\mathcal{E}(-B(t))dt + C\Big]\mathcal{F}(B) \tag{25}$$

where C is $n \times m$ constant matrix.

Proof. Let $U = \mathcal{E}(A) \cdot W \cdot \mathcal{F}(B)$, then

$$\frac{d}{dx}U = A(x)U + UB(x) + \mathcal{E}(A)\frac{d}{dx}W \cdot \mathcal{F}(B) \tag{26}$$

So Eq.(24) could be reduced to

$$\mathcal{E}(A)\frac{d}{dx}W \cdot \mathcal{F}(B) = P \tag{27}$$

or,

$$\frac{d}{dx}W = \mathcal{F}(-A) \cdot P \cdot \mathcal{E}(-B) \tag{28}$$

It's obviously that

$$W(x) = \int_0^x \mathcal{F}[-A(t)]P(t) \cdot \mathcal{E}[-B(t)]dt + C \tag{29}$$

C is $n \times m$ constant matrix . □

4. Solutions of Riccati equation

In mathematical investigation of the dynamics of a system, the introduction of a nonlinearity always leads to some form of the Riccati equation [Watkins]:

$$\frac{d}{dx}y + a(x)y^2 + b(x)y + c(x) = 0 \tag{30}$$

But it is usually the case that not even one solution of the Riccati equation is known. In the following text, we try to give out solutions of Riccati equation in matrix form:

$$\frac{d}{dx}W + WPW + WB - AW - Q = 0 \tag{31}$$

where $A(x) = (a_{ij})_{nn}$, $B(x) = (b_{ij})_{mm}$, $P(x) = (p_{ij})_{mn}$, $Q(x) = (q_{ij})_{nm}$.

4.1. Proof of Theorem.2.2

Proof. 1. Firstly , define[Polyanin, Ch 0.1.4]

$$W_2 := \mathcal{E}(PW + B) \tag{32}$$

so W_2 is reversible, if $PW + B$ is bounded;
meanwhile,

$$\frac{d}{dx}W_2 = (PW + B)W_2 \tag{33}$$

Secondly, let $W_1 := WW_2$, so

$$\tfrac{d}{dx}W_1 \quad = \tfrac{d}{dx}W \cdot W_2 + W \cdot \tfrac{d}{dx}W_2 = \tfrac{d}{dx}W \cdot W_2 + W \cdot \Big[PW + B\Big]W_2 = \Big[\tfrac{d}{dx}W + WPW + WB\Big]W_2 \tag{34}$$

so, with Eq.(31) and Definition (32), it holds

$$\frac{d}{dx}W_1 = AW_1 + QW_2 \tag{35}$$

Take the relationship (33) and (35) into consideration,

$$\frac{d}{dx}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ P & B \end{bmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{36}$$

we can solve W_1 and W_2.

On the other hand, according to Definition (32) , it's obviously that

$$W_2|_{x=0} = \mathcal{E}(PW + B)|_{x=0} = I \tag{37}$$

so it goes without saying that

$$W_1|_{x=0} = (WW_2)|_{x=0} = W|_{x=0} \tag{38}$$

We immediately obtain

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathcal{E}\left(\begin{bmatrix} A & Q \\ P & B \end{bmatrix}\right) \cdot \begin{bmatrix} W\,|_{x=0} \\ I \end{bmatrix} \tag{39}$$

Therefore $W = W_1 \cdot W_2^{-1}$ is the solution of Eq.(31).

2. Similarly, we can get

$$\frac{d}{dx}\begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} I & W\,|_{x=0} \end{bmatrix} \cdot \begin{bmatrix} -B & P \\ Q & -A \end{bmatrix} \tag{40}$$

so, $W = U_2^{-1} \cdot U_1$ is also the solution of Eq.(31).

3. But the two solutions are equivalence! That is,

$$W_1 \cdot W_2^{-1} = U_2^{-1} U_1 \tag{41}$$

or

$$U_2 \cdot W_1 - U_1 \cdot W_2 = 0 \tag{42}$$

Because, according to Eq.(36) and Eq.(40)

$$\begin{aligned} & \frac{d}{dx}\left[U_2 \cdot W_1 - U_1 \cdot W_2\right] = \frac{d}{dx}U_2 \cdot W_1 + U_2 \cdot \frac{d}{dx}W_1 - \frac{d}{dx}U_1 \cdot W_2 - U_1 \cdot \frac{d}{dx}W_2 \\ = \; & \left[U_1P - U_2A\right] \cdot W_1 + U_2 \cdot \left[AW_1 + QW_2\right] - \left[U_2Q - U_1B\right] \cdot W_2 - U_1 \cdot \left[PW_1 + BW_2\right] = 0 \end{aligned} \tag{43}$$

As a result,

$$U_2 \cdot W_1 - U_1 \cdot W_2 = const. = [U_2 \cdot W_1 - U_1 \cdot W_2]\big|_{x=0} = 0 \tag{44}$$

which implied that two solutions are equivalence.

4. Uniqueness. If Eq.(31) has more than one solution,such as $X(x), Y(x)$, under the same initial condition,i.e. $X(0) = Y(0)$. Let $W(x) = X(x) - Y(x)$. So it is clear that what we need to prove is equitant to show

$$\begin{cases} \frac{d}{dx}W + WPW + WB - AW = 0 \\ W|_{x=0} = 0 \end{cases} \tag{45}$$

has uniqueness solution $W(x) = 0$.

Take advantage the proof steps we have established: according to step(39) and (35),

Any solution of Eq.(45), such as $W(x)$, it is reasonable to define

$$W_2 = \mathcal{E}(PW + B), \qquad W_1 = W \cdot W_2$$

It follows that W_2 is bounded ,

$$\frac{d}{dx}W_1 = AW_1 \tag{46}$$

and

$$W_1 = \mathcal{E}[A] \cdot W_1|_{x=0} = \mathcal{E}[A] \cdot W|_{x=0} = 0 \tag{47}$$

Therefore, $W(x) = W_1 \cdot W_2^{-1} = 0$

□

4.2. *Simplify solutions of Riccati equation by particular solution*

In the research of Riccati equation, particular solution plays crucial important role. Too much of works have been done. The first important result in the analysis of the Riccati equation is that if one solution is known then a whole family of solutions can be found [Watkins].

Theorem 4.1. *The same conditions as theorem 2.2, Riccati equation*

$$\frac{d}{dx}W + WPW + WB - AW - Q = 0 \tag{48}$$

has the unique solution

$$W = Y + \mathcal{E}(A - YP) \cdot (W|_{x=0}) \cdot \left[I + \int_0^x R(t)\,(t)\,dt \cdot (W\,|_{x=0})\right]^{-1} \cdot \mathcal{F}(-[B + PY]) \tag{49}$$

where

$$\begin{cases} Y \text{ is solution of Eq.(48) when } W|_{x=0} = 0, \text{i.e. } Y|_{x=0} = 0 \\ R := \mathcal{F}(-[B + PY]) \cdot P \cdot \mathcal{E}(A - YP) \end{cases} \tag{50}$$

Proof. 1. According to Theorem 2.2 , Eq.(48) has solutions. Take any one of it, such as Y, and let

$$V = W - Y \tag{51}$$

It follows that

$$\begin{aligned} VPV &= (W - Y)P(W - Y) = (WPW - YPY) - (W - Y)PY - YP(W - Y) = (WPW - YPY) - VPY - YPV \\ &\overset{Eq.(48)}{=} \left([-\tfrac{d}{dx}W - WB + AW + Q] - [-\tfrac{d}{dx}Y - YB + AY + Q]\right) - VPY - YPV \\ &= \left(-\tfrac{d}{dx}V + AV - VB\right) - VPY - YPV = -\tfrac{d}{dx}V + (A - YP)V - V(B + PY) \end{aligned} \tag{52}$$

That is,

$$\frac{d}{dx}V + VPV + V(B + PY) - (A - YP)V = 0 \tag{53}$$

2. Obviously, $\mathcal{E}\big(A - YP\big)$ and $\mathcal{F}\big(-[B + PY]\big)$ are reversible , we may let

$$V = \mathcal{E}\big(A - YP\big) \cdot U \cdot \mathcal{F}\big(-[B + PY]\big) \tag{54}$$

Now Eq.(53) could be transformed into

$$\Big[\mathcal{E}\big(A - YP\big) \cdot \frac{d}{dx} U \cdot \mathcal{F}\big(-[B + PY]\big) + (A - YP)V - V(B + PY)\Big] + VPV + V(B + PY) - (A - YP)V = 0 \tag{55}$$

or,

$$\frac{d}{dx} U + U\Big[\mathcal{F}\big(-[B + PY]\big) \cdot P \cdot \mathcal{E}\big(A - YP\big) \cdot \Big]U = 0 \tag{56}$$

3. Let

$$R := \mathcal{F}\big(-[B + PY]\big) \cdot P \cdot \mathcal{E}\big(A - YP\big) \tag{57}$$

According to Theorem 2.2, U has solution

$$U = W_1 \cdot W_2^{-1} \tag{58}$$

where

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathcal{E}\left(\begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} U\,|_{x=0} \\ I \end{bmatrix} = \left(I + \int_0^x \begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix} dt\right) \cdot \begin{bmatrix} U\,|_{x=0} \\ I \end{bmatrix} = \begin{bmatrix} U\,|_{x=0} \\ I + \int_0^x R(t)\,(t)\,dt \cdot U\,|_{x=0} \end{bmatrix} \tag{59}$$

Now, Let's consider how to choose Y , so that both W and $U\,|_{x=0}$ are as simple as possible. It's clear that

$$\text{when } Y|_{x=0} = 0,\ U|_{x=0} = Y|_{x=0} = W|_{x=0}$$

In this case,

$$U = W|_{x=0} \cdot \left[I + \int_0^x R(t)\,(t)\,dt \cdot (W\,|_{x=0})\right]^{-1} \tag{60}$$

It should be noticed that $\left[I + \int_0^x R(t)\,(t)\,dt \cdot (W\,|_{x=0})\right]$ is reversible, otherwise

$$I + \int_0^x R(t)\,(t)\,dt \cdot (W\,|_{x=0}) \equiv 0 \tag{61}$$

which is clearly impossible.

According to transformation(54), the solution of Eq.(48) is

$$W = Y + V = Y + \mathcal{E}\big(A - YP\big) \cdot W|_{x=0} \cdot \left[I + \int_0^x R(t)\,(t)\,dt \cdot (W\,|_{x=0})\right]^{-1} \cdot \mathcal{F}\big(-[B + PY]\big) \tag{62}$$

where $Y(x) \equiv 0$ *, if and only if* $Q(x) \equiv 0$.

□

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